MATH4210: Financial Mathematics Tutorial 9

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26 March, 2024

Continuous Market Models

Question (a)

We consider a continuous time market, where the interest rate $\boxed{r = 0}$, *and the risky asset* $S = (S_t)_{0 \le t \le T}$ *follows the Black-Scholes model with initial value* $S_0 = 1$, *drift* μ *and volatility* $\sigma > 0$ *(without any dividend), so that*

$$S_t = S_0 \exp\left(\boxed{\mu} - \sigma^2/2\right)t + \sigma B_t \quad \text{modelled under } \mathbb{P}$$

Solve the following questions:

(a) A self-financing portfolio is given by (x, ϕ), *where* x *represents the initial wealth of the portfolio, and* ϕ_t *represents the number of risky asset in the portfolio at time* t. *Let* $\Pi_t^{x,\phi}$ *be the wealth process of the portfolio, write down the dynamic of* $\Pi^{x,\phi}$ *in* $t \in [0, T]$ *in form of*

$$\boxed{d\Pi_t^{x,\phi} = \alpha_t dt + \beta_t dB_t.}$$

α, β are adapted to $\mathbb{F}$

Find α *and* β.

Continuous Market Models

Denote $\hat{\pi}_t = e^{-rt}\pi_t$ and $\hat{S}_t = e^{-rt}S_t$ $\boxed{d\hat{\pi}_t = \phi_t d\hat{S}_t}$ $(*)$

$= \pi_t$ $= S_t$

$d\pi_t = \boxed{d\hat{\pi}_t = \phi_t d\hat{S}_t} = \phi_t dS_t = \phi_t(\mu S_t dt + \sigma S_t dB_t) = \overbrace{\mu\phi_t S_t}^{\alpha_t} dt + \overbrace{\sigma\phi_t S_t}^{\beta_t} dB_t$

Definition (Self-financing Portfolio)

(Slides 4B) We say the portfolio (Π_t) is self-financing if

$$d\Pi_t = \overbrace{\boxed{(\Pi_t - \phi_t S_t)}}^{\text{risk-free asset}} r dt + \underbrace{\phi_t dS_t}_{\text{risky asset}}.$$

Question (b)

(b) There exists a unique risky-neutral probability $\mathbb{Q}$, together with a Brownian motion $B^{\mathbb{Q}}$ under the probability measure $\mathbb{Q}$. Give the expression of S_t as a function of $(t, B_t^{\mathbb{Q}})$.

$S_t = S_0 \exp((r - \frac{\sigma^2}{2})t + \sigma B_t^{\mathbb{Q}})$.

Since $r = 0$, we have: $S_t = S_0 \exp(-\frac{\sigma^2}{2}t + \sigma B_t^{\mathbb{Q}})$.

$V_0 = \mathbb{E}^{Q}[S_T^2]$

$S_T = \exp(-\frac{\sigma^2}{2}T + \sigma B_T^{Q})$

$\Rightarrow V_0 = \mathbb{E}^{Q}[\exp(-\sigma^2 T + 2\sigma B_T^{Q})]$

$= e^{-\sigma^2 T}\mathbb{E}^{Q}[e^{2\sigma B_T^{Q}}]$

(Moment generating characteristic) $= e^{-\sigma^2 T} \cdot e^{\frac{1}{2}\cdot(2\sigma)^2 \cdot T} = e^{\sigma^2 T}$.

Question (c)

(c) We first consider a derivative option with payoff $g(S_T) = S_T^2$ *at maturity* T.
(i) Compute the value

$$V_0 = \mathbb{E}^{\mathbb{Q}}[S_T^2].$$

if $r \neq 0$: $\quad V_0 = \mathbb{E}^{Q}[e^{-rT}g(S_T) \mid S_0 = 1]$

What is the option price at $0 \leq t \leq T$:

$$V_t = \mathbb{E}^{Q}[e^{-r(T-t)}g(S_T) \mid S_t]$$

$V(t,x) = x^2 e^{\sigma^2(T-t)}$

$\partial_t V(t,x) = -\sigma^2 x^2 e^{\sigma^2(T-t)}$

$\partial_x V(t,x) = 2x e^{\sigma^2(T-t)}$

$\partial^2_{xx} V(t,x) = 2 e^{\sigma^2(T-t)}$

$\partial_t V(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} V(t,x) = 0$

$V(T,x) = x^2 e^{\sigma^2(T-T)} = x^2$

Question (c)

(ii) Let $v(t,x) := x^2 \exp \sigma^2(T-t)$, compute $\partial_t v$, $\partial_x v$ and $\partial^2_{xx} v$. Check that v satisfies the equation

$$\partial_t v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} v(t,x) = 0, \qquad v(T,x) = x^2.$$

① $V(t,x)$ is $\mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} g(S_T) \mid S_t = x]$

② For PDE $\partial_t V(t,x) + b(t,x)\partial_x V(t,x) + \frac{1}{2}\sigma(t,x)^2 \partial^2_{xx} V(t,x) - rV(t,x) = 0$ with $V(T,x) = g(x)$, then $\mathbb{E}[e^{-r(T-t)} g(X_T) \mid X_t = x]$ solves the PDE where $dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t$.

(Feynman-Kac formula)

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$V(t, S_t) = V(t, S_0 e^{-\sigma^2/2\, t + \sigma B_t^{\Theta}})$

$= u(t, B_t^{\Theta})$

$dv(t, S_t) = du(t, B_t^{\Theta})$. Ito's formula.

$= \partial_t u(t, B_t^{\Theta}) dt + \partial_x u(t, B_t^{\Theta}) dB_t$

$+ \frac{1}{2} \partial_{xx}^2 u(t, B_t^{\Theta}) dt$

Question (c)

(iii) Remember that S_t is a function of (t, B_t^{Θ}), apply the Ito formula on $v(t, S_t)$ to deduce that $S_t = S_0 e^{-\frac{\sigma^2}{2} t + \sigma B_t^{\Theta}}$

$V(t, S_t) = S_t^2 \cdot e^{\sigma^2(T-t)} = S_0^2 e^{-\sigma^2 t + 2\sigma B_t} \cdot e^{\sigma^2 T - \sigma^2 t} = e^{\sigma^2 T} \cdot e^{-2\sigma^2 t + 2\sigma B_t}$

$$S_T^2 = V_0 + \int_0^T \underline{\phi_t dS_t}, \quad where \quad \phi_t := \partial_x v(t, x).$$

Then deduce that V_0 is the (no-arbitrage) price of the derivative option $g(S_T) = S_T^2$.

$u(t, B_t) = V(t, S_t) = S_t^2 e^{\sigma^2(T-t)}$

$= e^{\sigma^2(T-t)} \left(e^{-\frac{\sigma^2}{2} t + \sigma B_t} \right)^2$

$= e^{\sigma^2(T-t)} e^{-\sigma^2 t + 2\sigma B_t} = e^{\sigma^2 T} e^{-2\sigma^2 t + 2\sigma B_t}$

$$\partial_t u = e^{\sigma^2 T} e^{-2\sigma^2 t + 2\sigma B_t} (-2\sigma^2)$$

$$\partial_x u(t, B_t) = e^{\sigma^2 T} \cdot e^{-2\sigma^2 t + 2\sigma B_t} (\cdot 2\sigma)$$

$$\partial_{xx}^2 u(t, B_t) = e^{\sigma^2 T} \cdot e^{-2\sigma^2 t + 2\sigma B_t} (2\sigma)^2$$

$$du(t, B_t) = e^{\sigma^2 T} e^{-2\sigma^2 t + 2\sigma B_t} \left(\cancel{-2\sigma^2 dt} + 2\sigma dB_t + \cancel{\tfrac{1}{2}(2\sigma)^2 \cdot dt}\right)$$

$$= e^{\sigma^2 T} \cdot 2\sigma \, e^{-2\sigma^2 t + 2\sigma B_t} \, dB_t \quad (*)$$

$$S_t = e^{-\frac{\sigma^2}{2} t + \sigma B_t} \Rightarrow dS_t = \sigma S_t dB_t$$

$$\left.\begin{array}{l} S_t = e^{-\frac{\sigma^2}{2} t + \sigma B_t} \end{array}\right\} \Rightarrow dS_t = \sigma \cdot e^{-\frac{\sigma^2}{2} t + \sigma B_t} dB_t$$

$$(*) = e^{\sigma^2 (T-t)} \cdot 2 \cdot S_t \cdot \left(\sigma e^{-\frac{\sigma^2}{2} t + \sigma B_t}\right) dB_t$$

$$= 2 e^{\sigma^2 (T-t)} S_t dS_t$$

$$dV = 2 e^{\sigma^2 (T-t)} S_t dS_t$$

$$\left.\begin{array}{l} V(T, S_T) - V(0, S_0) \doteq \int_0^T \underline{2 e^{\sigma^2 (T-t)}} S_t dS_t \\ V(T, S_T) = S_T^2, \quad V(0, S_0) = \underbrace{e^{\sigma^2 T}}_{\substack{= \\ V_0 \text{ in previous question}}} \end{array}\right\} \Rightarrow \boxed{S_T^2 = V_0 + \int_0^T \underline{\phi_t} dS_t}$$

V_t is a self-financing portfolio.

$$dV_t = (\cdots) \underbrace{r}_{=0} dt + \phi_t dS_t$$

Also, V_t replicate the cash-flow of S_T^2

Then by no-arbitrage approach: $V_0 = V(0, S_0)$ is the option price at time 0.

But $V(0, S_0) = e^{\sigma^2 T}$, it implies, V_0 is the option price at time 0.

Continuous Market Models

Proposition (Ito for diffusion process)

For an diffusion process $dX_t = \mu_t dt + \sigma_t dB_t$ *and a function* $f \in C^{1,2}$*, then*

processes

$$f(t, X_t) = f(0, X_0) + \int_0^t (\partial_t f + \partial_x f)(u, X_u) dX_u + \int_0^t \frac{1}{2}\partial^2_{xx} f(u, X_u) \boxed{d[X]_u},$$

where $\boxed{d[X]_u = \sigma_t^2 dt.}$

$dt : 1$

$dB_t : 0.5$

" $(dB)^2$ " $= d[B]_t : 1.$

$dt \cdot dB_t : 1.5$

Question (d)

(d) We now consider another option with (path-dependent) payoff $\int_0^T S_t^2 dt.$

$f(x) := x^2.$ $f(S_T) = f(S_0) + \int_0^T \partial_x f(S_t) dS_t + \int_0^T \frac{1}{2}\partial^2_{xx} f(S_t) d[S]_t$

(i) Apply the Ito formula to deduce that

$= S_0^2 + 2\int_0^T S_t dS_t + \cancel{\frac{1}{2}}\cancel{2}\int_0^T \cdot\, d[S]_t$

$$S_T^2 = S_0^2 + \int_0^T 2S_t dS_t + \sigma^2 \int_0^T S_t^2 dt.$$

$d[S_t] = \sigma^2 S_t^2 dt$

$\to S_T^2 = S_0^2 + \int_0^T 2S_t dt + \int_0^T \sigma^2 S_t^2 dt.$

Continuous Market Models

Question (d)

(ii) From the above, one obtains that

payoff .

$$\sigma^2 \boxed{\int_0^T S_t^2 dt} = S_T^2 - S_0^2 - \int_0^T 2S_t dS_t.$$

Deduce the replication cost and replication strategy of the derivative option $\int_0^T S_t^2 dt$. (Hint: Use the above replication strategy for the option $g(S_T) = S_T^2$.)

$$\int_0^T S_t^2 dt = \frac{1}{\sigma^2}\left(S_T^2 - 1 - \int_0^T 2S_t dS_t\right).$$

$$= \frac{1}{\sigma^2}\left(V_0 + \int_0^T \phi_t dS_t - 1 - \int_0^T 2S_t dS_t\right) = \boxed{\frac{V_0 - 1}{\sigma^2}} + \int_0^T \boxed{\psi_t} dS_t$$

$$= \frac{1}{\sigma^2}\left(e^{\sigma^2 T} - 1\right) = E^{Q}\left[\int_0^T S_t^2 dt\right]$$

where $\psi_t = \frac{1}{\sigma^2}(\phi_t - 2S_t)$